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HOW BLACK HOLES GET THEIR KICKS: GRAVITATIONAL RADIATION RECOIL REVISITED

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ABSTRACT

Gravitational waves from the coalescence of binary black holes carry away linear momentum, causing center of mass recoil. This "radiation rocket" effect has important implications for systems with escape speeds of order the recoil velocity. We revisit this problem using black hole perturbation theory, treating the binary as a test mass spiraling into a spinning hole. For extreme mass ratios ($q \equiv m_1/m_2 \ll 1$) we compute the recoil for the slow inspiral epoch of binary coalescence very accurately; these results can be extrapolated to $q \sim 0.4$ with modest accuracy. Although the recoil from the final plunge contributes significantly to the final recoil, we are only able to make crude estimates of its magnitude. We find that the recoil can easily reach $\sim 100-200\,\mathrm{km/s}$, but most likely does not exceed $\sim 500\,\mathrm{km/s}$. Though much lower than previous estimates, this recoil is large enough to have important astrophysical consequences. These include the ejection of black holes from globular clusters, dwarf galaxies, and high-redshift dark matter halos.

Subject headings: black hole physics—gravitation—gravitational waves — galaxies: nuclei

1. INTRODUCTION AND BACKGROUND

Along with energy and angular momentum, gravitational waves (GWs) carry *linear* momentum away from a radiating source (Bonnor & Rotenberg 1961; Peres 1962; Bekenstein 1973). Global conservation of momentum requires that the center of mass (COM) of the system recoil. This recoil is independent of the system's total mass.

Fitchett (1983) first computed GW recoil for binaries. He treated the members as non-spinning point masses (m_1, m_2), the gravitational force as Newtonian, and included only the lowest GW multipoles needed for momentum ejection. For circular orbits Fitchett's recoil is

$$V_F \simeq 1480\,\mathrm{km/s}\,\frac{f(q)}{f_{\max}}\left(\frac{2GM/c^2}{r_{\rm term}}\right)^4 , \qquad (1)$$

where $r_{\rm term}$ is the orbital separation where GW emission terminates, $q = m_1/m_2 \le 1$ is the mass ratio, and $M = m_1 + m_2$ is the total mass. The function $f(q) = q^2(1-q)/(1+q)^5$ has a maximum $f_{\max}$ at $q \simeq 0.38$, is zero for $q = 1$, and has the limit $f(q) \approx q^2$ for $q \ll 1$.

Equation (1) tells us that in the coalescence of binary black holes (BHs)—where $r_{\rm term}$ can approach GM/c^2—the kick might reach thousands of km/s. This is far greater than the escape velocity of many globular clusters (typically ~ 30 km/s), and may even exceed galactic escape velocities (~ 1000 km/s). Recoil could thus have important astrophysical implications (Redmount & Rees 1989) [some of which are explored in a companion paper (Merritt et al. 2004; Paper II)]. This has motivated us to revisit this problem.

Equation (1) indicates that the recoil is strongest at small separations, when the relativistic effects neglected by Fitchett are most important. This issue has been addressed in restricted circumstances using perturbation theory (Nakamura & Haugan 1983; Fitchett & Detweiler 1984; Nakamura, Oohara, & Kojima 1987), post-Newtonian expansions (Wiseman 1992; Kidder 1995), and numerical relativity (Andrade & Price 1997; Anninos & Brandt 1998; Brandt & Anninos 1999; Lousto & Price 2004). Unlike previous analyses, our treatment applies to the strong-gravity, fast-motion regime around spinning holes undergoing binary coalescence. Using BH perturbation theory we model the dynamics of the binary, the generation of GWs, and the backreaction of those waves on the system up to the inner-most stable circular orbit (ISCO). Our results are accurate only for extreme mass ratio inspirals ($q \ll 1$), but we can extrapolate to $q \sim 0.4$ with modest error. We model the GW emission from the final plunge more crudely.

2. OVERVIEW OF GRAVITATIONAL RADIATION RECOIL

The rate at which momentum is radiated is given by

$$\frac{dP^k_{\rm GW}}{dt} = \frac{r^2}{16\pi}\int d\Omega \left\langle \dot{h}^2_+ + \dot{h}^2_\times \right\rangle n^k , \qquad (2)$$

where $h_{+,\times}$ are the "plus" and "cross" GW polarizations, n^k is a unit radial vector from the source, and r is the distance to the observer (Thorne 1980). [We have set $G = c = 1$; an overdot refers to a derivative with respect to coordinate time t; angle brackets denote averaging over several wavelengths.] The binary's COM recoil is $dP^k_{\rm COM}/dt = -dP^k_{\rm GW}/dt$.

Decomposing $h_{+,\times}$ into multipoles in the wave zone (Thorne 1980), Eq. (2) can be expanded (to lowest order) as

$$\frac{dP^k_{\rm GW}}{dt} = \frac{2}{63}\left\langle \frac{d^4\mathcal{I}^{ijk}}{dt^4}\frac{d^3\mathcal{I}^{ij}}{dt^3}\right\rangle + \frac{16}{45}\left\langle \epsilon^{kpq}\frac{d^3\mathcal{I}^{pj}}{dt^3}\frac{d^3\mathcal{S}^{qj}}{dt^3}\right\rangle , \qquad (3)$$

where $\mathcal{I}^{ij}$, $\mathcal{S}^{ij}$, and $\mathcal{I}^{ijk}$ are the symmetric, trace-free mass quadrupole, current quadrupole, and mass octupole moments. Recoil thus arises from "beating" between different multipoles. Applying Eq. (3) to a Newtonian binary and integrating yields Eq. (1).

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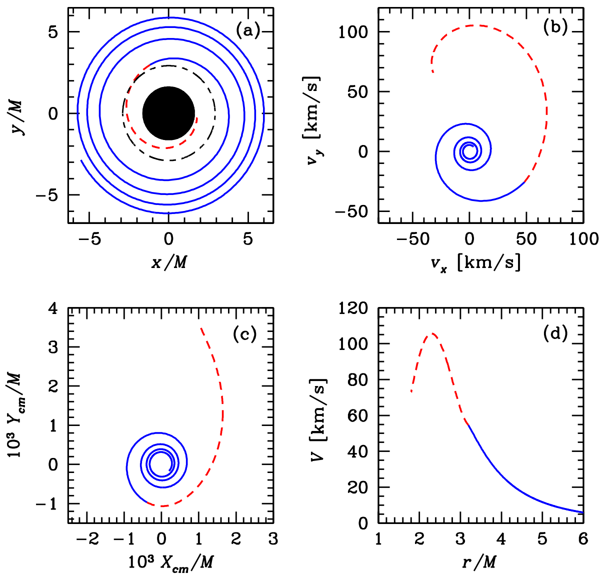

FIG. 1.— Recoil from prograde coalescence with $a/M = 0.8$, $\eta = 0.1$ ($q = 0.127$). Solid (blue) lines represent quantities during the inspiral, as calculated using our Teukolsky equation solver. Dashed (red) lines are calculations during the plunge (using the "upper-limit" prescription discussed in section 4). The plunge is truncated shortly before the particle enters the event horizon. The different panels are: (a) Orbit of the mass μ about the central spinning hole. The dashed circle is the location of the ISCO. (b) Recoil velocity of the center of mass. The spiral ends when GW emission is cut off. (c) Motion of the binary's center of mass. (d) Total center of mass recoil velocity, $(v_x^2 + v_y^2)^{1/2}$.

Wiseman (1992) provides an intuitive description of the recoil: When two non-spinning bodies are in circular orbit, the lighter mass moves faster and is more effective at "forward beaming" its radiation. Net momentum is ejected in the direction of the lighter mass's velocity, with opposing COM recoil. When $m_1 = m_2$, the beaming is symmetric and the recoil vanishes. The instantaneous recoil continually changes direction over a circular orbit, so the COM traces a circle. Neglecting radiation reaction, this circle closes, and the recoil averages to zero over each orbit. With radiative losses, the orbit does not close, and the recoil accumulates. This accumulation proceeds until the holes plunge and merge, shutting off the radiated momentum flux and yielding a net, non-zero kick velocity (cf. Fig. 1).

Spin complicates this picture by breaking the binary's symmetry. Consider an equal-mass binary, with one member spinning parallel to the orbital angular momentum. Due to spin-induced frame dragging, the non-spinning body's speed—and hence radiation beaming—is enhanced. Kidder (1995) has treated this spin-orbit interaction in post-Newtonian theory. Specializing his Eq. (3.31) to a circular, non-precessing orbit, the total kick for two bodies with spins $\mathbf{S}_{1,2} = \tilde{a}_{1,2} m_{1,2}^2 \hat{\mathbf{z}}$ parallel (or antiparallel) to the orbital angular momentum is

$$V_{\rm kick} = \left| V_F + 883\,{\rm km/s}\,\frac{f_{\rm SO}(q,\tilde{a}_1,\tilde{a}_2)}{f_{\rm SO,\,max}} \left(\frac{2M}{r_{\rm term}}\right)^{9/2} \right| , \tag{4}$$

where the spin-orbit scaling function $f_{\rm SO}(q,\tilde{a}_1,\tilde{a}_2) = q^2(\tilde{a}_2 - q\tilde{a}_1)/(1+q)^5$. The "correction" causes significant recoil even when $q = 1$ (and hence $V_F = 0$). The spin-orbit term is largest when $q = 1$ and the spins are maximal and antiparallel ($\tilde{a}_1 = -\tilde{a}_2 = \pm 1$; $f_{\rm SO,\,max} \equiv 1/16$). The recoil vanishes for $q = 1$ and spins equal and parallel ($\tilde{a}_1 = \tilde{a}_2$)—a symmetric binary.

Since we work in the $q \ll 1$ limit, we ignore the smaller body's spin, which incurs an error $\sim q^2\tilde{a}_1$ in the orbital dynamics (Kidder 1995). Our extreme mass ratio analysis treats the binary in an effective-one-body sense: a non-spinning point particle with mass $\mu = m_1 m_2/M$ orbits a Kerr hole with mass $M = m_1 + m_2$ and spin $\mathbf{S} = \tilde{a}M^2\hat{\mathbf{z}}$. There is an ambiguity, however, in how one translates the physical spin parameter $\tilde{a}_2$ of the hole to the "effective" spin parameter $\tilde{a}$. Damour (2001) provides a relation between these parameters, valid in the post-Newtonian limit for $\tilde{a} < 0.3$: $\tilde{a} = \tilde{a}_2(1 + 3q/4)/(1+q)^2$. Because of this ambiguity, we present our results in terms of the effective-spin-parameter $\tilde{a}$. Even if the larger hole's spin is nearly maximal ($\tilde{a}_2 \simeq \pm 1$), finite mass ratios $q \gtrsim 0.1$ restrict our results to spins with $|\tilde{a}| \lesssim 0.8 - 0.9$.

When applied to a perturbation calculation of the head-on collision of two BHs, an effective-one-body scaling of the GW energy flux ($\dot{E}_{\rm GW} \propto q^2$) in which $q \to \eta = \mu/M = q/(1+q)^2$ has been shown to agree with results from full numerical relativity (Smarr 1978). We use a similar "scaling up" procedure for the momentum flux: In perturbation theory $\dot{P}^j_{\rm GW} \propto q^2$. We then substitute $q^2 \to f(q)$ (Fitchett & Detweiler 1984). [In terms of η, the scaling function is given by $f(q) \to f(\eta) = \eta^2\sqrt{1-4\eta}$, and is maximized at $\eta = 1/5$.] Using $f(q)$ [or $f(\eta)$] to scale the momentum flux assumes both bodies are non-spinning and that the orbit is quasi-circular. For simplicity, approximate spin corrections to $f(q)$ based on Eq. (4) are ignored (incurring errors $\lesssim 30\%$ if $q \lesssim 0.4$) (cf. Paper II).

3. INSPIRAL RECOIL FROM PERTURBATION THEORY

Our model binary consists of a mass μ in circular, equatorial orbit about a BH with mass M and effective spin $a = \tilde{a}M$. (GWs rapidly reduce eccentricity, so circularity is a good assumption for many astrophysical binaries.) When $\mu \ll M$, binary evolution is well described using BH perturbation theory (Teukolsky 1973). We treat the binary's spacetime as a Kerr BH plus corrections from solving the perturbed Einstein equations—the Teukolsky equation. Specifically, we solve a linear wave equation for the complex scalar function Ψ_4, which describes radiative perturbations to the hole's curvature. Far from the binary, $\Psi_4 = (\ddot{h}_+ - i\ddot{h}_\times)/2$; it therefore encodes information about the GW fields in the distant wave zone, as well as the energy, momentum, and angular momentum carried by those fields.

Far from the binary Ψ_4 has the expansion:

$$\Psi_4 = \frac{1}{r}\sum_{lm} Z_{lm} S_{lm}(\theta; a\omega_m) e^{im\phi - i\omega_m t_R} \,. \tag{5}$$

In terms of Boyer-Lindquist coordinates (t, r, θ, ϕ), $t_R = t - r$ is retarded time, $\omega_m = m\Omega_{\rm orb}$ is a harmonic of the orbital frequency, $S_{lm}(\theta; a\omega_m)$ is a spheroidal harmonic, and Z_{lm} is a complex number found by solving a particular ordinary differential equation (Hughes 2000).

The linear momentum flux can be extracted by combining Eqs. (2) and (5). The resulting expression is simplest in the "corotating" frame, $\phi^{\rm corot} = \phi(t) - \Omega_{\rm orb}t$:

$$\dot{\mathcal{P}}_{\rm GW} = \frac{1}{2}\sum_{ll'm} \frac{Z_{lm}\bar{Z}_{l'(m+1)}}{\omega_m\omega_{m+1}} \int_0^\pi S_{lm}S_{l'(m+1)}\sin^2\theta d\theta \; . \tag{6}$$

Here, $\dot{\mathcal{P}}_{\rm GW} = e^{-i\phi(t)}[\dot{P}^x_{\rm GW} + i\dot{P}^y_{\rm GW}]$, and an overbar denotes complex conjugation. Similar expressions give the energy and angular momentum fluxes. The recoil velocity is found by integrating Eq. (6), starting at initial time T_0 when the binary is at large separation [and the recoil is well described by Eq. (1)], and ending at time T when GW emission terminates:

$$v_x + iv_y = -\frac{1}{M}\int_{T_0}^T e^{i\phi(t)}\dot{\mathcal{P}}_{\rm GW}\, dt \; . \tag{7}$$

Our procedure starts with a point-source on a circular geodesic orbit with specified energy E and angular momentum L_z. Solving the Teukolsky equation gives us the energy, momentum, and angular momentum fluxes of GWs to infinity and down the event horizon. (The linear momentum flux down the horizon does not affect the recoil.) In the adiabatic limit (in which GW backreaction changes the orbit very slowly, $r/\dot{r} \ll 2\pi/\Omega_{\rm orb}$), the energy and angular momentum fluxes $(\dot{E}_{\rm GW}, \dot{L}_{z,{\rm GW}})$ are used to evolve to a new geodesic with $E - \dot{E}_{\rm GW}\Delta t$ and $L_z - \dot{L}_{z,{\rm GW}}\Delta t$. Repeating this procedure for a sequence of geodesics generates a slow inspiral trajectory. The momentum flux along this trajectory and associated recoil velocity are then calculated via Eqs. (6) and (7).

This prescription can be used to calculate the recoil velocity only up to the ISCO. There the slow, adiabatic inspiral of the particle transitions to a rapid "plunge" that terminates when the particle crosses the event horizon (cf. Fig. 1a). Our Fourier decomposition of Ψ_4 is no longer valid as there are no well-defined harmonics ω_m for plunging trajectories.

Figure 2a shows the perturbation theory calculation of the ISCO recoil for a binary with reduced mass ratio $\eta = 0.1$ ($q = 0.127$). The solid curve in Figure 1a can be fit by

$$V_{\rm isco} = 422\,{\rm km/s}\,\frac{f(q)}{f_{\rm max}}\left(\frac{2M}{r_{\rm isco}}\right)^{2.63+0.06r_{\rm isco}/M} \quad , \tag{8}$$

where $r_{\rm isco}$ is the spin-dependent ISCO radius [defined for $q = 0$ by Eq. 2.21 of Bardeen, Press, & Teukolsky (1972)], and we have included the appropriate scaling function (valid for $q \lesssim 0.4$). Although our adiabatic assumption is violated for $\eta = 0.1$ (especially for large, prograde spins) our results are still valid since $V_{\rm isco}/f(q)$ is only weakly dependent on q (and is independent of q in the $q \to 0$ limit).

For retrograde orbits around rapidly spinning holes, the ISCO is at large radius ($9M$ for $\tilde{a} = -1$) and Fitchett's Newtonian formula [Eq. (1)] agrees well with our result. For prograde inspiral into rapidly spinning holes, the ISCO is deep in the strong field, where relativistic effects become important and suppress the recoil relative to Fitchett's result.

4. RECOIL ESTIMATES FROM THE FINAL PLUNGE

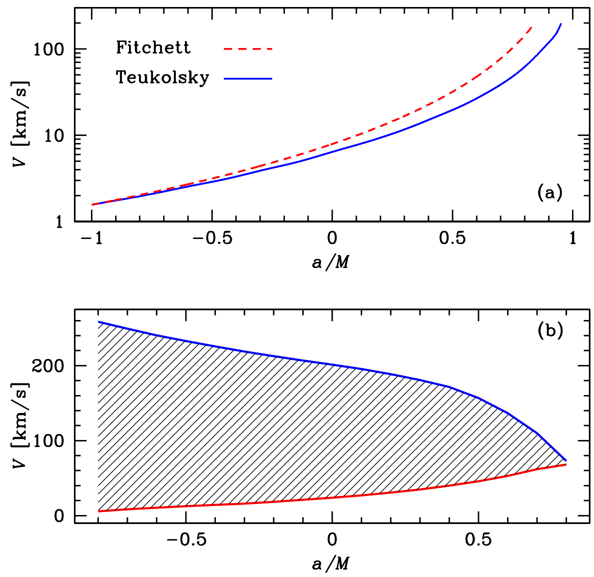

FIG. 2.— Recoil velocity versus effective spin a/M for $\eta = 0.1$ ($q = 0.127$). (a) Recoil velocity up to the ISCO. The solid (blue) curve is our Teukolsky equation result. The dashed (red) curve shows the Newtonian recoil prediction [Eq. (1)], which is substantially higher for large, prograde spins (smaller ISCO radius). (b) Upper and lower limits for the total recoil. The shaded region represents our uncertainty in the final kick velocity. The detailed shape of the upper-limit curve depends on the nature of our truncated-power-law ansatz.

During the plunge, the small body's motion is dominated by the Kerr effective potential rather than radiation-reaction forces (Ori & Thorne 2000). It is easy to match a plunging geodesic with constant E and L_z onto an inspiral trajectory near the ISCO. With a code that does not Fourier expand Ψ_4 (Khanna 2003; Martel 2003), one could properly compute the GW emission and associated recoil along such a plunging trajectory (when $q \ll 1$).

Since we do not have such a code at hand, we must estimate the wave emission more crudely. Our results from the inspiral show that, for a given spin, $\dot{\mathcal{P}}_{\rm GW}$ is well described by a power law in radius, $\dot{\mathcal{P}}_{\rm GW} \propto r^{-\alpha}$, from large r up to the ISCO. As an approximate "upper limit" of the recoil, we make the ansatz that this power law can be continued past the ISCO. This must break down at some point: the power-law reflects the circularity of the inspiral orbit and should be suppressed by the increasingly radial motion during the plunge. To prevent the momentum flux from diverging, we truncate the power law at $3M$, replacing it with the condition that $(dt/d\tau)\dot{\mathcal{P}}_{\rm GW}$ = constant, where τ is proper time along the plunge geodesic. This allows the momentum flux to "redshift away" as the particle approaches the horizon. Using the recoil velocity at the ISCO as initial conditions [Sec. 3] and a plunge trajectory with coordinates $[r(t), \phi(t)]$, we use Eq. (7) and our truncated-power-law ansatz to compute the accumulated recoil until a cutoff time T when the horizons of the holes come into contact (in a quasi-Newtonian interpretation of the coordinates). The upper-curve of Figure 2b shows the result of this calculation (for $\eta = 0.1$).

We also perform a separate "lower-limit" calculation. A plunge trajectory is computed as before, but in place of the power-law ansatz for $\dot{\mathcal{P}}_{\rm GW}$, we integrate the truncated, multipole expansion of Eq. (3) instead. In this calculation the momentum flux initially grows like a power law, but then decreases as the plunging trajectory nears the event horizon. Because we neglect higher multipoles (which are extremely important in the fast-motion, strong-gravity region), this method likely underestimates the recoil. The total accumulated recoil at the cutoff time T using this method is shown in the lower curve of Figure 2b (also for $\eta = 0.1$).

The shaded region between the two curves in Figure 2b represents our uncertainty in the total recoil at the end of the plunge. This uncertainty is largest for retrograde orbits around rapidly spinning holes, in which the distance the particle must "plunge" is greatest. For prograde inspiral into rapidly spinning holes, much of the recoil is due to emission during the slow inspiral phase, for which our BH perturbation techniques are well-suited. Figure 1 shows the relative contributions from the inspiral and plunge for such a scenario.

Although the two calculations for the plunge recoil give rather different results, useful astrophysical information is contained in the approximate upper and lower bounds that they represent. The estimate $V \sim 120\,\mathrm{km/s}$ bisects the shaded region of Figure 2b and represents a typical recoil velocity for this mass ratio. Note also that the numbers in Figure 2 can be scaled to higher mass ratios by multiplying by $f(q)/f(\eta = 0.1)$. For $q \approx 0.38$ this implies that our results can be augmented by a factor ≈ 2.3.

5. DISCUSSION

The punchline of this analysis is simple: quasi-Newtonian estimates have significantly overestimated the kick velocity from anisotropic GW emission during binary coalescence. The recoil is strongest when the smaller member is deep in the strong-field of the large black hole. General relativistic effects, such as the gravitational redshift and spacetime curvature-scattering, act on the emitted GWs and reduce the recoil.

Though reduced, the recoil remains large enough to have important astrophysical consequences. Recoils with $V \sim 10$–$100\,\mathrm{km/s}$ are likely; kicks of a few hundred km/s are not unexpected; and the largest possible recoils are probably $\lesssim 500\,\mathrm{km/s}$. These speeds are smaller than most galactic escape velocities, suggesting that BH mergers that follow galaxy mergers will remain within their host structures. However, these recoils are similar to the escape speeds of dwarf galaxies; and they may be sufficient to escape from mergers in high redshift structures [$z \gtrsim 5 - 10$; cf. Barkana & Loeb (2001), Fig. 8]. Binary BH ejection from globular clusters is quite likely, with significant implications for the formation of intermediate mass black holes (IMBH) via hierarchical mergers (Miller & Colbert 2003). Our recoil estimates will also be useful in simulations of supermassive and IMBH evolution in dark halos (Volonteri, Haardt, & Madau 2003; Madau et al. 2004).

Future papers will present the formalism used for this analysis, and will investigate the influence of orbital inclination on the recoil. More work in perturbation theory also remains in addressing the recoil from the plunge and final ringdown of the merging black holes.

Finally, Redmount & Rees (1989) have speculated that spin-orbit misalignment could lead to recoil directed out of the orbital plane. This recoil might accumulate secularly rather than oscillate, and would be similar to the "electromagnetic rocket" in pulsars with off-centered magnetic dipole moments (Harrison & Tademaru 1975; Lai, Chernoff, & Cordes 2001). We suspect that this effect occurs but it is likely small compared to the recoil from the final plunge and merger. Firm estimates of the final kick velocity will rely on correctly modelling the final phase of BH coalescence. For comparable mass binaries, full numerical relativity will ultimately be needed to accurately compute the GW recoil.

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